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USE OF EIGENVALUE TECHNIQUE IN FINITE ELEMENT TIDAL COMPUTATIONS

B. M. PATIL[1] AND B. V. RAO[2]*

[1]*Central Water and Power Research Station, Pune 411 024, India*
[2]*Department of Civil Engineering, Indian Institute of Technology, Bombay 400 076, India*

SUMMARY

This paper presents the results of some studies on the development and application of a finite element method (FEM) with a closed-form solution technique for time discretization. The closed-form solution is based on the eigenvalues/vectors of a coefficient matrix. The method is first applied to the one-dimensional linearized shallow water equations and then extended to the two-dimensional shallow water equations. An attempt is made to improve its efficiency by incorporating time splitting and using the closed-form solution technique only for linear terms. Some case studies of a rectangular channel and harbour are presented to illustrate the satisfactory working of the method. © 1997 by John Wiley & Sons, Ltd. Int. j. numer. methods fluids 24: 953–963, 1997.

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KEY WORDS: coastal hydrodynamics; shallow water equations; finite element method; finite difference method

1. INTRODUCTION

For the hydrodynamic modelling of wide coastal water bodies, the two-dimensional shallow water equations must be solved numerically. A number of finite difference and finite element models for this purpose are available in the literature. In general the finite element models[1–8] are based on discretization of the time domain by finite differences. An attempt is made here to use a closed-form solution technique based on the eigenvalues/vectors of a coefficient matrix for the time domain, while the space domain is discretized by finite elements. The formulation of this FEM to solve the 1D (simplified) and 2D shallow water equations and its applications are described below.

2. GOVERNING EQUATIONS

2.1. 1D shallow water equations

The following linearized 1D shallow water equations with constant water depth and without a friction term have been chosen to test the scheme, since these have analytical solutions:[2]

$$\frac{\partial z}{\partial t} + h\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial z}{\partial x} = 0, \tag{2}$$

* Correspondence to: B. V. Rao, Department of Civil Engineering, Indian Institute of Technology, Bombay 400 076, India

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where z is the water level, h is the water depth, u is the longitudinal flow velocity, g is the acceleration due to gravity, x is the distance along the channel and t is time.

These equations can be used to simulate a standing wave in a rectangular prismatic channel of length L, with constant water depth, closed at one end ($x = L$) and subjected to wave action at the open end ($x = 0$). The wave equation considered is

$$z = a' \sin(\omega t), \tag{3}$$

where a' is the amplitude, $\omega = 2\pi/T$ is the frequency of oscillation and T is the period. The analytical solutions[2,3] to the standing wave of equations (1) and (2) are

$$z = Ah \cos\Phi \sin(\omega t), \tag{4}$$

$$u = -Ac \sin\Phi \cos(\omega, t), \tag{5}$$

with $A = a'/h \cos(L/c)$, $\Phi = L(x/L - 1)/c$ and $c^2 = gh$.

2.2. 2D shallow water equations

The following two-dimensional, vertically integrated shallow water equations have been solved numerically by the FEM:

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial z}{\partial x} + fv - \tau u + a\nabla^2 u, \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial z}{\partial y} - fu - \tau v + a\nabla^2 v, \tag{8}$$

with $\tau = g(u^2 + v^2)^{1/2}/c^2 H$, where x and y are Cartesian co-ordinates, t is time, u and v are vertically averaged velocities in the x- and y-direction respectively, z is the water level above the reference plane, $H = h + z$ is the total water depth, f is the Coriolis parameter, a is the eddy viscosity and τ is the friction parameter.

3. FEM FORMULATION

3.1. 1D Eigenvalue FEM

Using the Galerkin method and linear shape functions $[N]$ for approximating z and u, equations (1) and (2) can be discretized to become the following for an element:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} 0 & C1 \\ M1 & 0 \end{bmatrix} \begin{Bmatrix} z \\ u \end{Bmatrix}, \tag{9}$$

where

$$M = \int [N]^{\mathrm{T}}[N]\,\mathrm{d}x, \qquad C1 = \int [N]^{\mathrm{T}}[N]\{h\}[N_{,x}]\,\mathrm{d}x, \qquad M1 = -g\int [N]^{\mathrm{T}}[N_{,x}]\,\mathrm{d}x.$$

After assembling all the element matrices, the global matrix equation can be written as

$$[AL]\{\dot{U}\} = [AR]\{U\} \tag{10}$$

or

$$\{\dot{U}\} = [K]\{U\}, \quad \text{with } [K] = [AL]^{-1}[AR]. \tag{11}$$

This matrix equation can be integrated directly[9] as

$$\{U\} = [X\,\mathrm{e}^{\Lambda T} X^{-1}]\{U_0\}, \tag{12}$$

where X is a matrix whose columns are the eigenvectors x_i of $[K]$, $e^{\Lambda T}$ is a matrix with $e^{\lambda_i t}$ on the diagonal (λ_i being eigenvalues of $[K]$) and zero elsewhere, X^{-1} is the inverse of X and $\{U_0\}$ is the vector of initial values of variables u_i.

In this FEM the solution is marched with a time step and Dirichlet-type boundary conditions are imposed by directly updating the boundary values after each step.

If n is the number of nodes, then $[AL]$, $[AR]$, $[K]$, $[X\,\mathrm{e}^{\Lambda T} X^{-1}]$, etc. are matrices of order $2n \times 2n$. The matrix $[AL]$ is tridiagonal for one-dimensional problems. The IMSL library routines available on the Cyber 840 computer have been used for computing X, X^{-1}, etc.

3.2. 2D eigenvalue FEM (EV)

The 1D eigenvalue FEM is extended to equations (6)–(8). Discretization of these equations by the Galerkin method using linear triangular elements leads to the matrix equation.

$$\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} C1 & C2 & C3 \\ MX1 & MX2 & MX3 \\ MY1 & MY2 & MY3 \end{bmatrix} \begin{Bmatrix} z \\ u \\ v \end{Bmatrix}. \tag{13}$$

The various submatrices of equation (13) are given below using the inner-product notation $\langle a, b \rangle = \int\int a \cdot b \, \mathrm{d}x \, \mathrm{d}y$:

$$\begin{aligned}
M &= \langle [N]^{\mathrm{T}}, [N] \rangle, \qquad C1 = CX1 + CX2, \\
C2 &= \langle [N_{,x}]^{\mathrm{T}}, [N]\{h\}[N \rangle], \qquad C3 = \langle [N_{,y}]^{\mathrm{T}}, [N]\{h\}[N] \rangle, \\
CX1 &= \langle [N_{,x}]^{\mathrm{T}}, [N]\{u\}[N \rangle], \qquad CX2 = \langle [N_{,y}]^{\mathrm{T}}, [N]\{v\}[N \rangle], \\
MX1 &= -\langle [N]^{\mathrm{T}}, g[N,x] \rangle, \qquad MY1 = -\langle [N]^{\mathrm{T}}, g[N,y \rangle], \\
MX2 &= MY3 = -(M2 + M3 + M4 - M5), \\
M2 &= \langle [N]^{\mathrm{T}}, [N]\{\tau\}[N]\{u\} \rangle, \qquad M3 = \langle [N]^{\mathrm{T}}, [N]\{u\}[N] \rangle, \\
M4 &= \langle [N]^{\mathrm{T}}, [N]\{v\}[N] \rangle, \qquad M5 = \langle [N]^{\mathrm{T}}, a\nabla^2] \rangle, \\
MX3 &= \langle [N]^{\mathrm{T}}, f[N] \rangle, \qquad MY2 = \langle [N]^{\mathrm{T}}, -f[N \rangle].
\end{aligned}$$

After assembly, equation (13) can be expressed in the form of equation (10) or (11) whose solution is given by equation (12).

In this case, if n is the number of nodes in the solution domain, then $[K]$ is a $3n \times 3n$ asymmetric matrix and $X, e^{\Lambda T}$ and X^{-1} are complex matrices.

Because of the non-linear terms in the continuity and momentum equations, the matrix $[K]$ is not stationary. Hence in each time step the matrix of eigenvectors, X, and its inverse must be computed, which consumes a large amount of CPU time.

3.3. Eigenvalue FEM with splitting (EVSP)

For efficient solution the EV method is improved to EVSP by using a splitting technique similar to the splitting method of Marchuk[10] and Chau and Lee.[11] In this case the shallow water equations can be split into two operators. This approach enables us to separate the linear and non-linear terms into two parts and use separate numerical solution algorithms.

Using the Galerkin method with linear triangular elements, the two-dimensional shallow water equations can be written in the form of equation (13) for an element. Further, this can be written as the matrix equation

$$\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 0 & C2 & C3 \\ MX1 & MX'2 & MX3 \\ MY1 & MY2 & MY'3 \end{bmatrix} \begin{Bmatrix} z \\ u \\ v \end{Bmatrix} + \begin{bmatrix} C1 & 0 & 0 \\ 0 & M''X2 & 0 \\ 0 & 0 & MY''3 \end{bmatrix} \begin{Bmatrix} z \\ u \\ v \end{Bmatrix}, \quad (14)$$

where

$$M'X2 = M'Y3 = M5, \qquad M''X2 - M''Y3 = -(M2 + M3 + M4).$$

After assembling over all the elements, equation (14) becomes

$$[AL]\{\dot{U}\} = [AR1]\{U\} + [AR2]\{U\}, \quad (15)$$

where $[AR1]$ incorporates linear terms and $[AR2]$ incorporates non-linear terms.

First we consider the linear part as follows:

$$[AL]\{\dot{U}\} = [AR1]\{U\} \quad (16)$$

or

$$\{\dot{U}\} = [K]\{U\}, \quad \text{with } [K] = [AL]^{-1}[AR1]. \quad (17)$$

The solution is

$$\{U\} = [X\mathrm{e}^{\Lambda T}X^{-1}]\{U_0\}. \quad (18)$$

Since $[K]$ is stationary, it is required to compute $[Xe^{\Lambda T}X^{-1}]$ once only at the beginning and this can be used in each time step, thus saving on CPU time.

The solution U obtained from the eigenvalue algorithm is used as initial condition for the next algorithm as follows. We have for an element

$$[AL]\{\dot{U}\} = [AR2]\{U\}. \quad (19)$$

Denoting $\bar{U}$ as the solution at time level $n+1$ and using the weighting coefficient Θ, we can obtain the equation

$$[AL - \Theta\Delta t AR2]\{U\} = [AL + (1-\Theta)\Delta t AR2]\{U\}, \quad (20)$$

which can be written as the system of equations

$$[A]\{\ddot{U}\} = \{R\}. \quad (21)$$

This can be solved using appropriate boundary conditions. From the form of $[AL]$ and $[AR2]$ it is seen that the system of equations is uncoupled for z, u and v and hence can be solved independently as n equations in n unknowns for each of z, u and v by an efficient band matrix solver. It is possible to iterate the solution for non-linear terms.

4. APPLICATIONS

4.1. 1D application

A channel with $L = 200$ m and $h = 4$ m is considered. It is discretized into four elements and five nodes as shown in Figure 1 with a constant element size of 50 m. The boundary conditions imposed

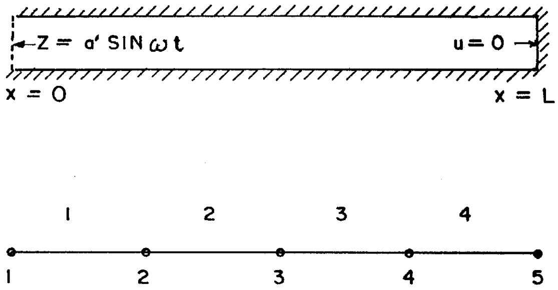

Figure 1. Rectangular channel and FEM mesh for 1D model

are $u = 0$ at the downstream closed end ($x = L$) and at the open end ($x = 0$) and the water level was prescribed by the sinusoidal boundary condition given by equation (3) with $a' = 0{\cdot}1$ m and period $T = 200$ s. The initial conditions were prescribed by the analytical formulae (4) and (5).

With these data, X, X^{-1} and $[X e^{\Lambda T} X^{-1}]$ are found to be dense matrices (fully populated). The eigenvalues and eigenvectors of the matrix $[K]$ are complex owing to its asymmetry. The complex parts in $[X e^{\Lambda T} X^{-1}]$ were less than 10^{-5}.

The model was operated with $\Delta t = 1$ s. The time histories (three cycles) of the water level at node 5 and the velocity at node 1 are shown in Figure 2 along with the analytical solutions. The figure indicates that the computed and analytical solutions are in good agreement. Thus the method works satisfactorily for $\Delta t = 1$ s and if the initial conditions are given by the analytical formulae.

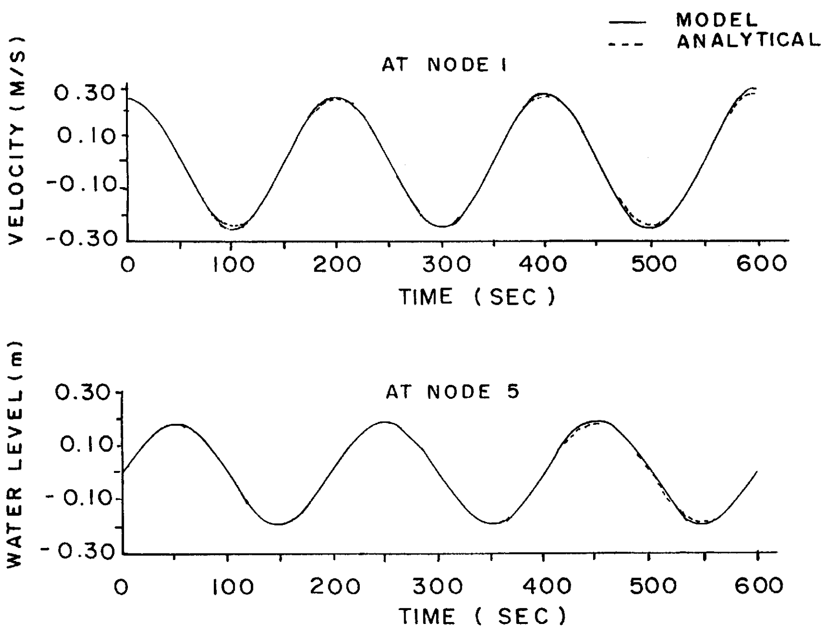

Figure 2. Water level and velocity time histories by 1D EV FEM ($\Delta t = 1$)

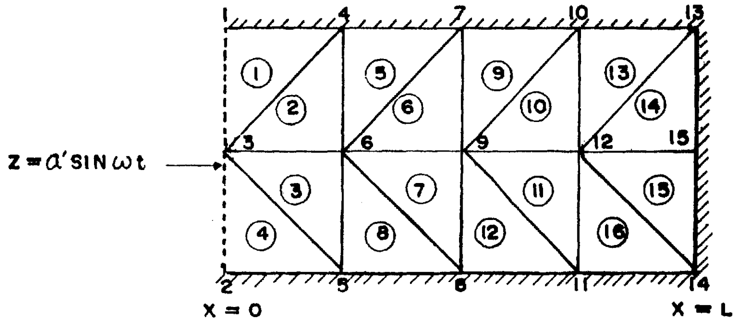

Figure 3. FEM mesh for rectangular channel for 2D model

4.2. 2D applications

Application to a rectangular channel. The EVSP FEM is tested by simulating a standing wave in a rectangular channel of width 100 m, $L = 200$ m, $h = 4$ m and with the mesh shown in Figure 3. The initial and boundary conditions used are the same as those of the 1D application. The friction and Coriolis terms are neglected. The water level at the closed end (node 15) and the velocity at the open end (node 3) are plotted along with the analytical solutions in Figure 4. They are obtained with $\Delta t = 2$ s and $a = 2$ m^2 s^{-1}. The corresponding results with $\Delta t = 4$ s and $a = 4$ m^2 s^{-1} are shown in Figure 5. These figures indicate good agreement between the computed and analytical solutions. Thus the method works satisfactorily for Δt less than 5 s and for a suitable value of a (to be decided by trial and error).

Application to a rectangular harbour. A simple problem of a rectangular harbour 20 km × 32·5 km in size has been considered. The depth of the harbour is assumed to be constant and equal to 10 m. The harbour is open at one end, as shown in Figure 6. A sinusoidal boundary condition given by equation (3) with $a' = 2$ m, $\omega = 2\pi/T$ and $T = 12{\cdot}4$ h is prescribed at the open boundary, as

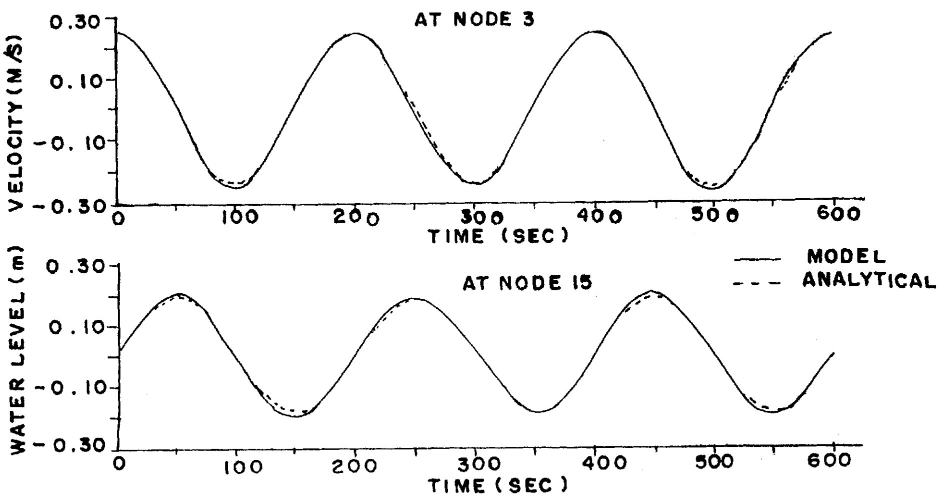

Figure 4. Water level and velocity time histories by EVSP ($\Delta t = 2, a = 2$) 2D rectangular channel model

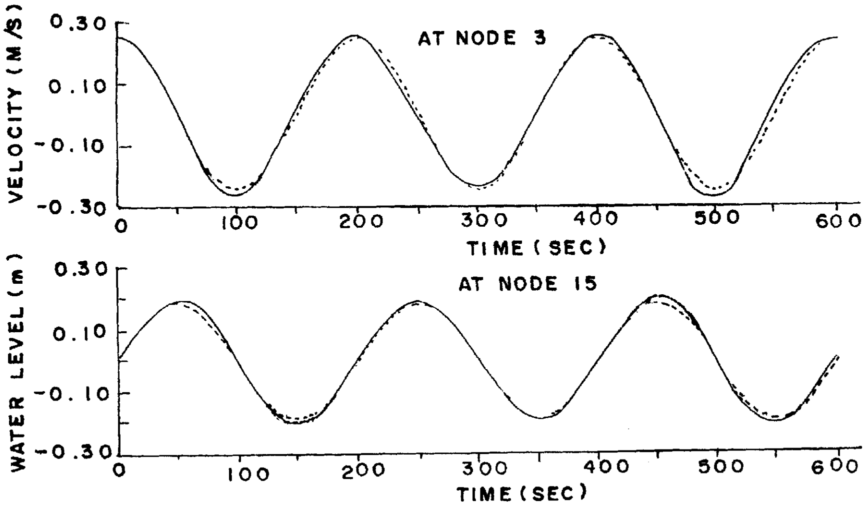

Figure 5. Water level and velocity time histories by EVSP ($\Delta t = 4, a = 4$) 2D rectangular channel model

shown in Figure 7. At the closed boundary the normal velocity is assumed to be zero. Chezy's coefficients and the Coriolis parameter are assumed to be 40 $m^{1/2}$ s^{-1} and 0·00005 respectively. The eddy viscosity coefficient is taken to be 4000 m^2 s^{-1} in the EV method. The FEM mesh used is shown in Figure 6(a) and contains 48 elements and 35 nodes. Both the EV and EVSP methods have been used to simulate the flow in the rectangular harbour with $\Delta t = 120$ s. Their results are compared with those obtained by well-tested methods, namely the ADI finite difference method[12] and the wave equation model (WEM).[5] The mesh used for the ADI finite difference method is shown in Figure 6(b), in which the water level and velocity are computed at different locations (staggered grid).

The four models EV, EVSP, ADI and WEM have been operated for the rectangular harbour problem. The models begin with a cold start and the results are stabilized after the second cycle. The results of the second cycle are used in the analysis. The time histories of the water level and velocity at node 7 obtained by all the methods are shown in Figures 8 and 9 respectively. From these figures it can be seen that the time histories obtained by the four methods are similar, with some phase difference. The water level variations of EV, EVSP and WEM are close, while that of ADI is a little different because of the larger phase difference and damping. The time histories of the velocity

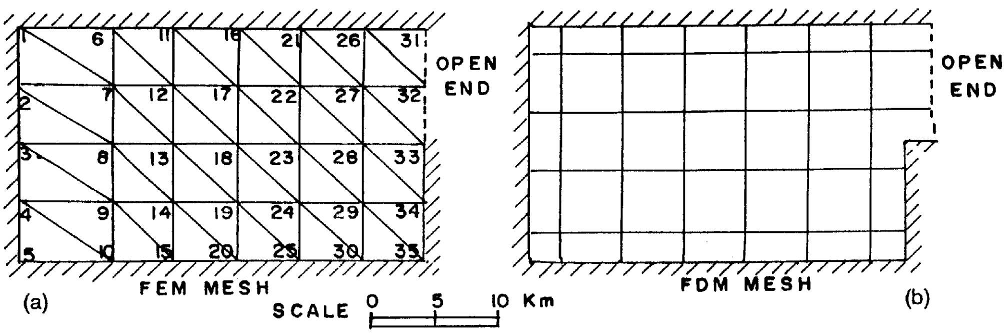

Figure 6. (a) FEM and (b) FDM meshes for 2D rectangular harbour model

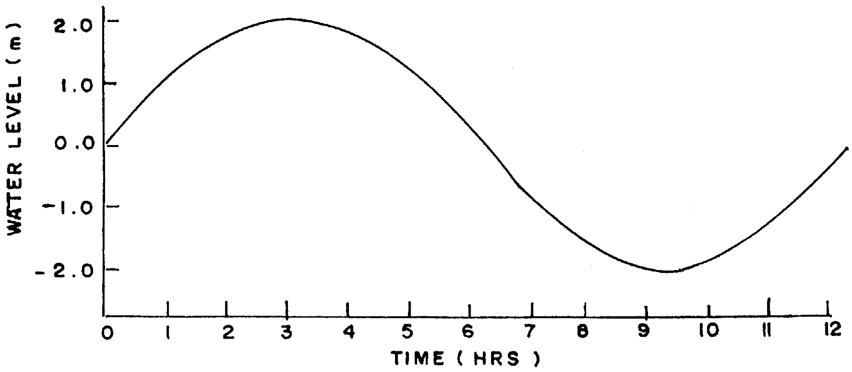

Figure 7. Boundary condition at open end for 2D rectangular harbour model

obtained by WEM and ADI are close, while those of EV and EVSP are a little different from that of WEM. The variations in the water level and velocity with the distance from the open end at a time of 3 h are shown in Figures 10 and 11 respectively, which indicate fewer spatial oscillations in the water level variations of EV and EVSP. In the velocity variation of ADI some oscillations are observed, while in the velocity variations of EV and EVSP there are fewer spatial oscillations. It can also be seen from these figures that the spatial variations in the water level and velocity obtained by EV and EVSP are very close to those of WEM, which is a rigorously tested method. This indicates satisfactory working of the EV and EVSP methods. However, it should be noted that these methods gave proper results when the eddy viscosity coefficient was taken to be 4000 $m^2 s^{-1}$, whereas this coefficient was neglected in the other two methods. The velocity field obtained by EVSP is shown in Figure 12.

In order to compare the efficiency of the various methods, the computer times required for running the various models on the Cyber 840A computer system are listed in Table I.

The table indicates that the ADI scheme allows the use of a larger time step and is the most efficient among the four. WEM, being implicit in the time domain, allows a larger time step than do EV and EVSP. Although the EV and EVSP methods are analytical but explicit in the time domain, they work for a smaller time step. In the EV method, since the solution involves the computation of eigenvalues and eigenvectors of a $3n \times 3n$ matrix and the inversion of a $3n \times 3n$ complex matrix in each step, it requires a very large computer time as indicated in Table I. Hence this method is not

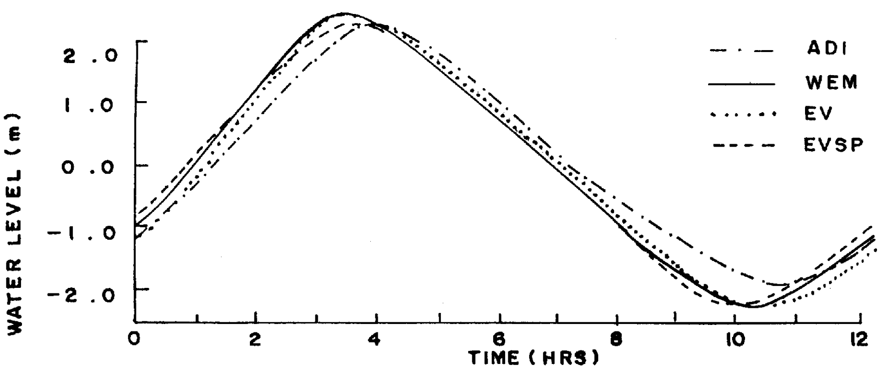

Figure 8. Time history of water level at node 7 (by ADI, WEM, EV, EVSP), 2D rectangular harbour model

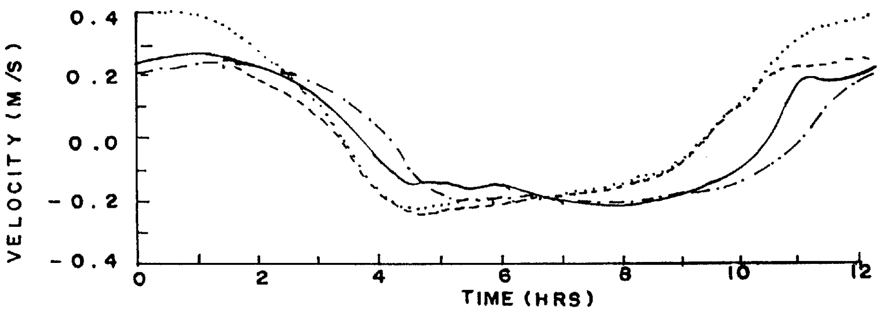

Figure 9. Time history of velocity at node 7 (by WEM, ADI, EV, EVSP), 2D rectangular harbour model

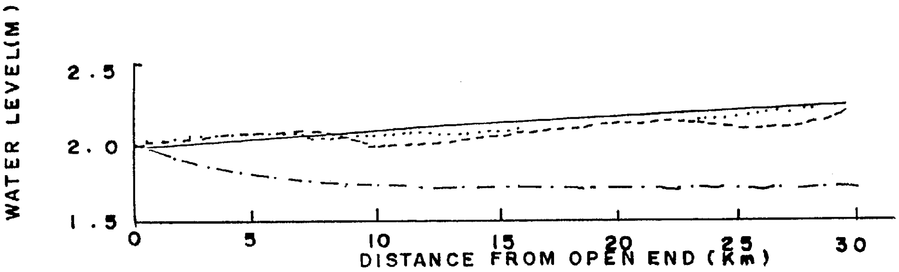

Figure 10. Water level variation with distance at 3 h (by ADI, WEM, EV, EVSP), 2D rectangular harbour model

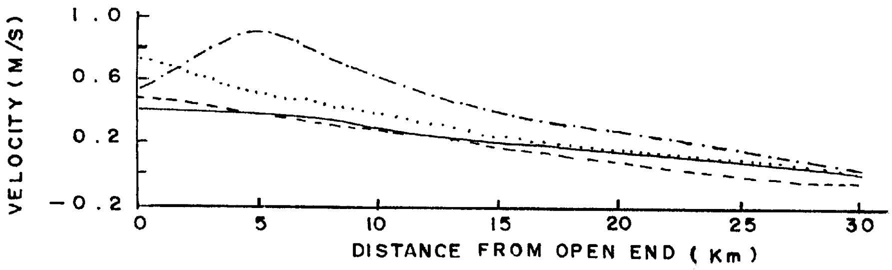

Figure 11. Velocity variation with distance at 3 h (by ADI, WEM, EV, EVSP), 2D rectangular harbour model

suitable for practical applications where a large number of nodes and a small time step are required to be used. The EVSP method is more efficient than the EV method, since it requires a computer time of 2650 s as against 27,830 s for the EV method for one cycle of 12·4 h. However, the performance of the EVSP method is poor compared with that of the ADI and WEM methods as seen from Table I.

5. CONCLUSIONS

In general, finite element methods for solution of the shallow water equations are based on discretization of the time domain by finite differences. An attempt is made here to develop and implement an FEM with a closed-form solution technique for time discretization, based on the eigenvalues/vectors of a coefficient matrix and using a finite element technique for space discretization. The method is used to solve the 1D (simplified) and 2D shallow water equations. Its

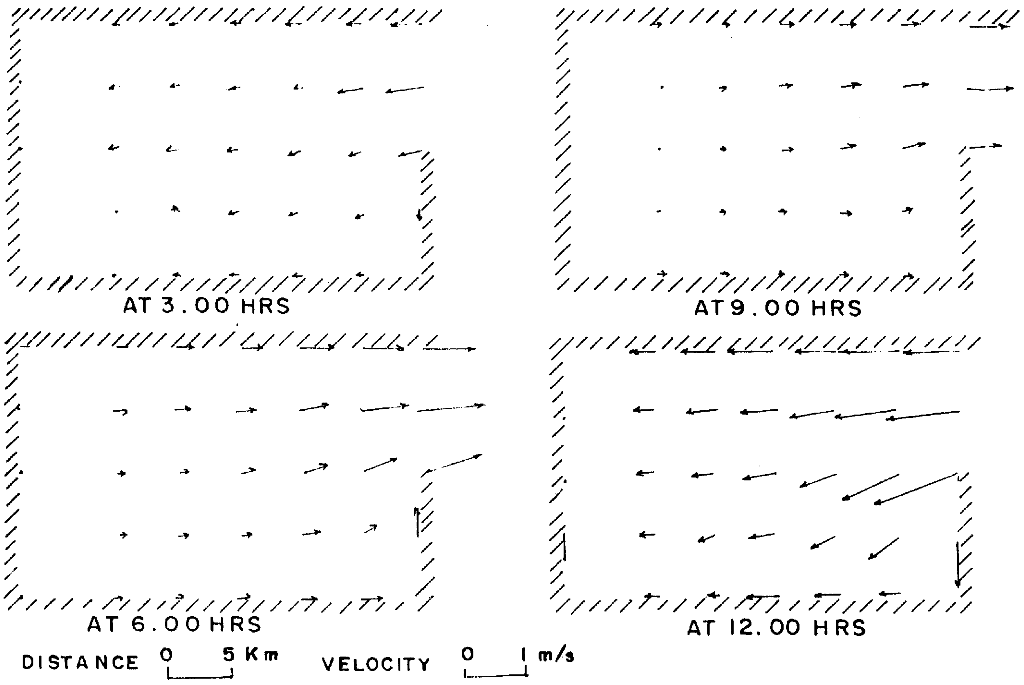

Figure 12. Velocity field by EVSP

Table I. Computer time required for rectangular harbour model

Method	Period (h)	Time step (s)	No. of time steps	No. of nodes	CPU time per period (s)	CPU time per step per node (10^{-3} s)
ADI (FDM)	12·4	720	62	27	2	1·2
WEM	12·4	360	124	35	25	5·8
EVSP	12·4	120	372	35	2650	203·5
EV	12·4	120	372	35	27830	2137·5

application to rectangular channel and harbour problems indicated satisfactory performance for a smaller time step and a suitable eddy viscosity. It is noticed that the efficiency of the eigenvalue FEM is improved by a factor of 10 if one uses time splitting and computes the eigenvalues/vectors of the coefficient matrix only once at the beginning. However, the performance is still poor compared to that of other methods.

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